Journal of Global Optimization **15:** 181–196, 1999.

Exceptional Families of Elements for Continuous Functions: Some Applications to Complementarity Theory

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(Received 20 July 1997; accepted in revised form 15 December 1998)

Abstract. Using the topological degree and the concept of exceptional family of elements for a continuous function, we prove a very general existence theorem for the nonlinear complementarity problem. This result is an alternative theorem. A generalization of Karamardian's condition and the asymptotic monotonicity are also introduced. Several applications of the main results are presented.

Key words: Exceptional family of elements, Complementarity problem and the fixed point theory

1. Introduction

Initially, a notion of "exceptional family of elements" for a continuous function was introduced in 1984 by T. E. Smith, using a special property of projection operator onto a closed convex set in the Euclidean space $(\mathbf{R}^n, <,>)$ [21]. Recently, using the topological degree, a more general notion of *exceptional family of elements* was introduced by G. Isac, V. Bulavski and V. V. Kalashnikov [11, 13]. Using this notion, in [11] are presented some *alternative existence theorems* for complementarity problems. A consequence of these results is the fact that, given a closed convex cone $\mathbf{K}$ in $\mathbf{R}^n$ and a continuous function $f : \mathbf{R}^n \to \mathbf{R}^n$, to conclude that the complementarity problem $CP(f, \mathbf{K})$ associated with f and $\mathbf{K}$ has a solution, it is sufficient to show that f is without exceptional families of elements with respect $\mathbf{K}$. It follows that it is interesting to know under what conditions a function is without exceptional families of elements with respect to a convex cone. This problem has been studied in [11–13, 25–27]. Now, in this paper we present some new conditions which imply that a function is without exceptional families of elements. As applications, we present a few existence theorems for complementarity problems, a generalization

⋆ The results presented in this paper were obtained when the first author was visiting professor (May-06-Jun 06-1997) at the University of Calabria (Cosenza, Italy) supported by the National Council of Research of Italy.

of Altman's fixed point theorem and an existence result for the complementarity problem associated to a P_0-function and applicable to the study of the solvability of the generalized complementarity problem in the sense of Cottle and Dantzig [3, 10, 20, 22]. We note that the complementarity theory has many and interesting applications in Optimization, Economics, Game Theory, Engineering, Mechanics etc. [2, 3, 6, 9, 13, 21, 24]. Finally, we note that the concept of exceptional families of elements recently, has been extended for variational inequalities in [25–27]. The results presented in this paper can be considered as a complementary part of the papers [11–13], [25–27].

2. Preliminaries

Let $(\mathbf{R}^n, <, >)$ be the Euclidean space and $\mathbf{K} \subset \mathbf{R}^n$ a closed pointed convex cone, i.e., $\mathbf{K}$ is a non-empty closed set satisfying the following properties:

k_1) $\mathbf{K} + \mathbf{K} \subseteq \mathbf{K}$

k_2) $\lambda\mathbf{K} \subseteq \mathbf{K}$ for all $\lambda \in \mathbf{R}_+$,

k_3) $\mathbf{K} \cap (-\mathbf{K}) = \{0\}$.

Whenever a closed pointed convex cone $\mathbf{K} \subset E$ is defined, we have an ordering on E defined by $x \leqslant y$, if and only if $y - x \in \mathbf{K}$. By definition the dual of $\mathbf{K}$ is

$$\mathbf{K}^* = \{y \in \mathbf{R}^n | \langle x, y\rangle \geqslant 0 \text{ for all } x \in \mathbf{K}\}$$

If $D \subset \mathbf{R}^n$ is a closed convex set we denote the projection onto D by P_D, that is, for every $x \in \mathbf{R}^n$, $P_D(x)$ is the unique element in D satisfying

$$\|x - P_D(x)\| = \min_{y\in D} \|x - y\|.$$

In particular if $\mathbf{K} \subset \mathbf{R}^n$ is a closed convex cone we denote the projection onto $\mathbf{K}$ by $P_{\mathbf{K}}$.

We recall that the projection $P_{\mathbf{K}}$ onto a closed convex cone $\mathbf{K}$ is characterized by the following properties. For every $x \in \mathbf{R}^n$, $P_{\mathbf{K}}(x)$ is the (unique) element in $\mathbf{K}$ satisfying the following conditions:

(i) $\langle P_K(x) - x, y\rangle \geqslant 0$ for all $y \in \mathbf{K}$,

(ii) $\langle P_K(x) - x, P_K(x)\rangle = 0$.

If $\mathbf{K}$ and $\mathbf{Q}$ are two closed convex cones in $\mathbf{R}^n$, we say that $\mathbf{K}$ and $\mathbf{Q}$ are *mutually polar* if $\mathbf{K} = \mathbf{Q}^0$, where $\mathbf{Q}^0$ is the *polar* of $\mathbf{Q}$, that is,

$$\mathbf{Q}^0 = \{x \in \mathbf{R}^n | \langle x, y\rangle \leqslant 0 \text{ for all } y \in \mathbf{Q}\}$$

We will use the following classical result.

THEOREM (Moreau [16]). *If* $\mathbf{K}$ *and* $\mathbf{Q}$ *are two mutually polar convex cones in the Euclidean space* $(\mathbf{R}^n, <, >)$ *and* $x, y, z \in \mathbf{R}^n$, *then the following statements are equivalent:*

(iii) $z = x + y$; $x \in \mathbf{K}$, $y \in \mathbf{Q}$ *and* $\langle x, y\rangle = 0$

(iv) $x = P_K(z)$ *and* $y = P_Q(z)$.

If $\mathbf{Q} = \mathbf{K}^0$, then by the *bipolarity theorem* it follows that $\mathbf{K} = \bar{\mathbf{K}} = \mathbf{Q}^0$ and hence $\mathbf{K}$ and $\mathbf{Q}$ are mutually polar.

By *Moreau's Theorem* each vector $z \in \mathbf{R}^n$ has a unique representation of the form

$$z = z^+ - z^- \tag{1}$$

where $z^+ = P_K(z)$ and $z^- = -P_{K^0}(z)$. (Note that $-z^-$ is the orthogonal complement of z^+).

We recall now the definition of the *general nonlinear complementarity problem.* Let $\mathbf{K} \subset \mathbf{R}^n$ be a pointed closed convex cone and $f : \mathbf{K} \to \mathbf{R}^n$, a function. The nonlinear complementarity problem associate with f and $\mathbf{K}$ is:

$$NCP(f, \mathbf{K}) : \begin{cases} \text{find } x_* \in \mathbf{K} \text{ such that} \\ f(x_*) \in \mathbf{K}^* \text{ and} \\ \langle x_*, f(x_*) \rangle = 0\,. \end{cases}$$

The existence of solution of this problem is not evident [3, 9, 19]. Because of this fact, many authors have proposed several kinds of existence theorems [2, 3, 5, 7–9, 14, 18, 19]. For the importance and the applications of the problem $NCP(f, \mathbf{K})$ the reader is referred to [2, 3, 6, 9, 13, 21, 22]. Finally, in this paper we will use the topological degree as it is presented in the book [15].

Let Ω be a bounded open subset of $\mathbf{R}^n$ and $y \in \mathbf{R}^n$ an arbitrary point. The closure Ω is written $\bar{\Omega}$ and its boundary $\partial\Omega$. We denote by $\mathcal{C}(\bar{\Omega})$ the linear space of continuous functions from $\bar{\Omega}$ into $\mathbf{R}^n$. If $F \in C(\bar{\Omega})$ and $y \in \mathbf{R}^n$ is such that $y \notin F(\partial\Omega)$, we denote by $\deg(F, \Omega, y)$ the *topological degree* associated with F, Ω and y. If $F, G \in \mathcal{C}(\bar{\Omega})$ we consider the homotopy $H(x, t) = tG(x) + (1 - t)F(x)$, $t \in [0, 1]$.

THEOREM (Poincaré–Böhl, [15]). *Let* $\Omega \subset \mathbf{R}^n$ *be an open bounded subset and* $F, G \in \mathcal{C}(\bar{\Omega})$ *two continuous mappings. If* $y \in \mathbf{R}^n$ *is an arbitrary point satisfying the condition*

$$y \notin \{H(x, t) | x \in \partial\Omega \text{ and } t \in [0, 1]\}$$

then we have the following equality, $\deg(G, \Omega, y) = deg(F, \Omega, y)$.

3. Main results

Let $(\mathbf{R}^n, \langle, \rangle)$ be the Euclidean space and $f : \mathbf{R}^n \to \mathbf{R}^n$ a continuous function. In the paper [11] are introduced the following notions.

We say that the family of points $\{x^k\}_{r>0} \subset R^h_+$ is an exceptional family of elements for f with respect to R^n_+ if $\|x^r\| \to +\infty$ as $r \to +\infty$, and for each $r > 0$ there exists $\mu_r > 0$ such that:

(f_1) $f_i(x^r) = -\mu_r x^r_i$ if $x^r_i > 0$

(f_2) $f_i(x^r) \geqslant 0$ if $x^r_i = 0$.

If, the cone $\mathbf{R}^n_+$ is replaced by an arbitrary cone $\mathbf{K} \subset \mathbf{R}^{n,}$, then we replace the notion defined above by the following:

We say that the family of points $\{x^r\}_{r>0} \subset \mathbf{R}^n$ is an exceptional family of elements for f, with respect to $\mathbf{K}$ if $\|(x^r)^+\| \to +\infty$ as $r \to +\infty$, and for each $r > 0$ the point $f((x^r)^+)$ belongs to the open ray

$$\mathcal{O}((x^r)^-; s_r) = \{y = (x^r)^- + \mu s_r | \mu > 0\}$$

where $s_r = (x^r)^- - (x^r)^+$.

These notions were studied in [11, 13] and generalized in [25–27]. Using the topological degree, in [11] and also in [13], it was proved that for any continuous function $f : \mathbf{R}^n \to \mathbf{R}^n$, there exists either a solution for the problem $NCP(f, \mathbf{R}^n_+)$ (respectively for $NCP(f, \mathbf{K})$), or an exceptional family of elements for f.

In his Habilitation Thesis [13], V.V. Kalashnikov introduced the following definition, for an exceptional family of elements, which is, in some sense, a unification of both previous definitions.

DEFINITION 1 [13].We say that the family of elements $\{x^r\}_{r>0} \subset \mathbf{K}$ is *an exceptional family of elements* for $f : \mathbf{R}^n \to \mathbf{R}^n$, with respect to the convex cone $\mathbf{K} \subset \mathbf{R}^n$, if and only if for every real number $r > 0$ there exists a real number $\mu_r > 0$ such that the vector $u_r = f(x^r) + \mu_r x^r$ satisfies the following conditions:

(e_1) $u_r \in \mathbf{K}^*$,

(e_2) $\langle u_r, x^r\rangle = 0$,

(e_3) $\|x^r\| \to +\infty$ as $r \to +\infty$.

We say that the exceptional family of elements $\{x^r\}_{r>0}$ for f is *regular* if for any $r > 0$, $\|x^r\| = r$. The next result was proved in [13] using the topological degree and the equivalence between the solvability of the problem $NCP(f, \mathbf{K})$ and the solvability of the nonlinear equation

$$f(P_K(x)) + x - P_K(x) = 0 \tag{2}$$

(known as the "normal equation").

For the same result, we will give now another proof, much more simple based on the equivalence between the solvability of the problem $NCP(f, \mathbf{K})$ and the solvability of the nonlinear equation

$$x - P_K(x - f(x)) = 0 \tag{3}$$

THEOREM 1. *For any continuous function $f : \mathbf{R}^n \to \mathbf{R}^n$, there exists, either a solution for the problem $NCP(f, \mathbf{K})$, or a regular exceptional family of elements for f with respect to* $\mathbf{K}$.

Proof. Consider the function

$$\Phi(x) = x - P_K(x - f(x)) \tag{4}$$

defined for any $x \in \mathbf{R}^n$. Using the properties (*i*) and (*ii*) of operator $P_{\mathbf{K}}$ we can show that the problem $NCP(f, \mathbf{K})$ has a solution if and only if the equation

$$\Phi(x) = 0 \tag{5}$$

is solvable. We use the following notations:

$$S_r = \{x \in \mathbf{R}^n | \|x\| = r\}, \ \ B_r = \{x \in \mathbf{R}^n | \|x\| < r\}$$

for any $r > 0$ and denote by I the identity mapping on $\mathbf{R}^n$. Consider the homotopy:

$$H(x, t) = tx + (1 - t)\Phi(x); 0 \leqslant t \leqslant 1. \tag{6}$$

From the definition of Φ we have

$$H(x, t) = x - (1 - t)P_K(x - f(x)); t \in [0, 1] \tag{7}$$

We use the topological degree and we apply the *Poincaré–Böhl Theorem* for $y = 0$ and $\Omega = B_r(\partial\Omega = S_r)$. We have the following two situations:

(A) There exists $r > 0$ such that $H(x, t) \neq 0$ for any $x \in S_r$ and any $t \in [0, 1]$. In this case by *Poincaré-Böhl Theorem* we have that $\deg(\Phi, B_r, 0) = \deg(I, B_r, 0)$. Since $\deg(I, B_r, 0) = 1$ we deduce that equation (5) has a solution in B_r, which implies that the problem $NCP(f, \mathbf{K})$ has a solution.

(B) For every $r > 0$ there exist $x^r \in S_r$ and $t_r \in [0, 1]$ such that

$$H(x^r, t_r) = 0. \tag{8}$$

If $t_r = 0$, from (6) we have that $\Phi(x^r) = 0$ and hence the problem $NCP(f, \mathbf{K})$ has a solution.

We also remark that t_r must be different from 1. Indeed, if $t_r = 1$, using again (6) we deduce that $x^r = 0$, which is impossible since $x^r \in S_r$. Hence, we can say that either the problem $NCP(f, \mathbf{K})$ has a solution or for any $r > 0$ there exists $x^r \in S_r$ and $t_r \in]0, 1[$ such that $H(x^r, t_r) = 0$. From (7) we have

$$x^r - (1 - t_r)P_K(x^r - f(x^r)) = 0 \tag{9}$$

or

$$\frac{1}{1 - t_r} x^r = P_K(x^r - f(x^r)). \tag{10}$$

Because $\mathbf{K}$ is a cone we have that $x^r \in \mathbf{K}$. Applying the properties (*i*) and (*ii*) of operator P_K we deduce,

$$\left\langle \frac{1}{1-t_r}x^r - (x^r - f(x^r)), y \right\rangle \geqslant 0 \text{ for all } y \in \mathbf{K}, \tag{11}$$

and

$$\left\langle \frac{1}{1-t_r}x^r - (x^r - f(x^r)), \frac{1}{1-t_r}\, x^r \right\rangle = 0. \tag{12}$$

If we put $\mu_r = t_r/1 - t_r$ in (11) and (12) we deduce

$$\langle \mu_r x^r + f(x^r), y\rangle \geqslant 0 \text{ for all } y \in \mathbf{K}, \tag{13}$$

and

$$\langle \mu_r x^r + f(x^r), x^r\rangle = 0. \tag{14}$$

Considering (13), (14) and the facts that for any $r > 0$, $x^r \in \mathbf{K}$ and $\|x^r\| = r$, we have that $\{x^r\}_{r>0}$ is a regular exceptional family of elements for f with respect to $\mathbf{K}$. □

REMARK. We observe that *Theorem 1* is valid even if f is defined only on the cone $\mathbf{K}$. Indeed, in this case we apply *Theorem 1* to the function $g : \mathbf{R}^n \to \mathbf{R}^n$ defined by $g(x) = f(P_K(x))$ for every $x \in \mathbf{R}^n$.

An immediate consequence of *Theorem 1* is the fact that if $f : \mathbf{K} \to \mathbf{R}^n$ is continuous and without exceptional families of elements with respect to $\mathbf{K}$, then the problem $NCP(f, \mathbf{K})$ is solvable.

DEFINITION 2. We say that $f : \mathbf{K} \to \mathbf{R}^n$ satisfies *condition* (θ) if there exists $\rho > 0$ such that for all x with $\|x\| > \rho$, there exists $y \in \mathbf{K}$ with $\|y\| < \|x\|$ such that $\langle x - y, f(x)\rangle \geqslant 0$.

THEOREM 2. *Let $f : \mathbf{K} \to \mathbf{R}^n$ be a continuous function. If f satisfies condition (θ), then it is without regular exceptional families of elements and the problem $NCP(f, \mathbf{K})$ has a solution.*

Proof. Suppose that f has a regular exceptional family of elements $\{x^r\}_{r>0} \subset \mathbf{K}$. We have

$$u_r = f(x^r) + \mu_r x^r \in \mathbf{K}^* \text{ for all } r > 0, \tag{15}$$

$$\langle x^r, u_r\rangle = 0 \text{ for all } r > 0, \tag{16}$$

and

$$\|x^r\| \to +\infty \text{ as } r \to +\infty \tag{17}$$

Take $r > 0$ such that $\|x^r\| > \rho$. Since f satisfies condition (θ), there exists $y_r \in \mathbf{K}$ such that $\|y_r\| < \|x^r\|$ and $\langle x^r - y_r, f(x^r)\rangle \geqslant 0$. We have

$$\begin{aligned} 0 &\leqslant \langle x^r - y_r, f(x^r)\rangle = \langle x^r - y_r, u_r - \mu_r x^r\rangle \\ &= \langle x^r - y_r, u_r\rangle - \mu_r \|x^r\|^2 + \mu_r \langle y_r, x^r\rangle \\ &\leqslant -\mu_r \|x^r\| [\|x^r\| - \|y_r\|] < 0, \end{aligned}$$

which is impossible. Hence, the function f is without regular exceptional families of elements with respect to $\mathbf{K}$ and applying *Theorem 1* we obtain the last conclusion of the theorem. □

Condition (θ) contains as a particular case the classical Karamardian's condition.

DEFINITION 3. [14] We say that $f : \mathbf{R}^n \to \mathbf{R}^n$ satisfies *Karamardian's condition* on $\mathbf{K}$ if there exists a closed bounded set $D \subset \mathbf{K}$ such that for all $x \in \mathbf{K} \backslash D$ there exists $y \in D$ such that $\langle x - y, f(X)\rangle \geqslant 0$.

PROPOSITION 3. *If $f : \mathbf{R}^n \to \mathbf{R}^n$ satisfies Karamardian's condition on $\mathbf{K}$ then f satisfies condition (θ).*

Proof. Let $D \subset \mathbf{K}$ be the set defined by Karamardian's condition. Since D is bounded, then there exists $\rho > 0$ such that $D \subset \bar{B}_\rho \cap \mathbf{K}$. For any x such that $\|x\| > \rho$ there exists an element $y \in D$ (that is such that $\|y\| \leqslant \rho < \|x\|$) verifying $x - y, f(x)\rangle \geqslant 0$. Hence condition (θ) is satisfied. □

Let $\varphi : [0, +\infty[\to [0, +\infty[$ be a function such that $\lim_{t \to +\infty} \varphi(t) = +\infty$ and $u \in \mathbf{K}$ an arbitrary element.

DEFINITION 4. We say that $f : \mathbf{K} \to \mathbf{R}^n$ is asymptotically (u, φ)-*monotone* if there exists a real number $\rho > 0$ (eventually sufficiently large) such that $\langle x - u, f(x) - f(u)\rangle \geqslant \|x - u\| \varphi(\|x - u\|)$ for all $x \in \mathbf{K}$ with $\|x\| > \rho$

PROPOSITION 4. *Any asymptotically (u, φ)-monotone operator $f : \mathbf{K} \to \mathbf{R}^n$ satisfies property (θ) with respect to $\mathbf{K}$.*

Proof. For every $x \in \mathbf{K}$ with $\|x\| > \max(\rho, \|u\|)$ we have

$$\langle x - u, f(x) - f(u)\rangle \geqslant \|x - u\| \varphi(\|x - u\|)$$

which implies

$$\langle x - u, f(x)\rangle \geqslant \langle x - u, f(u)\rangle + \|x - u\| \varphi(\|x - u\|).$$

Since $\|x\| > \|u\|$ we have $\|x - u\| > 0$ and

$$\langle x - u, f(x)\rangle \geqslant \|x - u\| \left[\left\langle \frac{x - u}{\|x - u\|}, f(u) \right\rangle + \varphi(\|x - u\|) \right].$$

Considering for u fixed, $f(u)$ as a continuous linear functional on $\mathbf{R}^n$ and applying *Weierstrass' Theorem* with respect to the compact set $S_1^+ = \{x \in \mathbf{K} | \|x\| = 1\}$, we deduce that there exists $\gamma \in \mathbf{R}$ such that $\langle x - u/\|x-u\|, f(u)\rangle \geqslant \gamma$ for any $x \in \mathbf{R}$ with $\|x\| > \max(\rho, \|u\|)$. Since $\lim_{t\to+\infty} \varphi(t) = +\infty$ we have that there exists $\rho_* > 0$ such that $\|x - u\| > \rho_*$ implies $\varphi(\|x-u\|) \geqslant -\gamma$, that is $\langle x - u, f(x)\rangle \geqslant 0$. If, for any $x \in \mathbf{K}$ satisfying $\|x\| > \max(\rho_* + \|u\|, \rho)$, we take $y = u$ we have immediately that f satisfies condition (θ) with respect to $\mathbf{K}$. □

Also, we may consider the following two generalizations of the (u, φ)-monotonicity.

DEFINITION 5. We say that $f : \mathbf{K} \to \mathbf{R}^n$ is *asymptotically* (u, v, φ)*-monotone*, if there exist $\rho > 0$ and $v \in \mathbf{K}$ such that $\langle x-u, f(x) - f(v)\rangle \geqslant \|x-u\|\varphi(\|x-u\|)$ for all $x \in \mathbf{K}$ with $\|x\| > \rho$.

DEFINITION 6. We say that $f : \mathbf{K} \to \mathbf{R}^n$ is *asymptotically* (u, g, φ)*-monotone*, if there exist $\rho > 0$ and a function $g : \mathbf{K} \to \mathbf{R}^n$ such that $\langle x - u, f(x) - g(u)\rangle \geqslant \|x-u\|\varphi(\|x-u\|)$ for all $x \in \mathbf{K}$ with $\|x\| > \rho$.

PROPOSITION 5. *If* $f : \mathbf{K} \to \mathbf{R}^n$ *is an asymptotically* (u, v, φ)*-monotone or* (u, g, φ)*-monotone function, then* f *satisfies property (*θ*) with respect to* $\mathbf{K}$.

Proof. The proof is similar to the proof of *Proposition 4* and we omit it. □

THEOREM 6. *If the function* $f : \mathbf{K} \to \mathbf{R}^n$ *is continuous and there exists* $\rho > 0$ *such that* $\langle x, f(x)\rangle \geqslant 0$ *for all* $x \in \mathbf{K}$ *with* $\|x\| > \rho$*, then* f *satisfies condition* (θ) *with respect to* $\mathbf{K}$ *and the problem* $NCP(f, \mathbf{K})$ *has a solution.*

Proof. We apply *Theorem 2* taking $y = 0$ for all $x \in \mathbf{K}$ with $\|x\| > \rho$. □

THEOREM 7. *If for the continuous function* $f : \mathbf{K} \to \mathbf{R}^n$ *there exists* $\rho > 0$ *such that for all* $x \in \mathbf{K}$ *with* $\|x\| = \rho$ *there exists* u *with* $\|u\| < \rho$ *such that* $\langle x - u, f(x)\rangle \geqslant 0$*, then the problem* $NCP(f, \mathbf{K})$ *has a solution.*

Proof. For all $x \in \mathbf{K}$ with $\|x\| > \rho$ denote by $T_\rho(x)$ the radial projection onto $S_\rho^+ = \{x \in \mathbf{K} | \|x\| = \rho\}$, i.e., $T_\rho(x) = \rho/\|x\|$. Consider the function $g : \mathbf{K} \to \mathbf{R}^n$ defined by

$$g(x) = \begin{cases} f(x), & \text{if } \|x\| \leqslant \rho \\ f(T_\rho(x)) + \|x - T_\rho(x)\| x, & \text{if } \|x\| > \rho. \end{cases}$$

For any $x \in \mathbf{K}$ with $\|x\| > \rho$ there exists $\lambda_x > 0$ such that $x = \lambda_x T_\rho(x)$. By assumption, for $T_\rho(x)$ there exists u_ρ^x with $\|u_\rho^x\| < \rho$ such that

$$\langle T_\rho(x) - u_\rho^x, f(T_\rho(x))\rangle \geqslant 0. \tag{18}$$

$$\begin{aligned}
\langle x - \lambda_x u_\rho^x, g(x)\rangle &= \langle \lambda_x T_\rho(x) - \lambda_x u_\rho^x, g(x)\rangle \\
&= \langle \lambda_x T_\rho(x) - \lambda_x u_\rho^x, f(T_\rho(x)) + \|x - T_\rho(x)\| x\rangle \\
&= \lambda_x \langle T_\rho(x) - u_\rho^x, f(T_\rho(x))\rangle + \|x - T_\rho(x)\|\|x\|^2 - \|x - T_\rho(x)\|\langle \lambda_x u_\rho^x, x\rangle \\
&\geqslant \|x - T_\rho(x)\|[\|x\|^2 - \lambda_x u_\rho^x, x\rangle] \\
&\geqslant \|x - T_\rho(x)\|[\lambda_x^2\|T_\rho(x)\|^2 - \lambda_x^2\|u_\rho^x\|\,\|T_\rho(x)\| \\
&= \|x - T_\rho(x)\|\lambda_x^2\|T_\rho(x)\|[\|T_\rho(x)\| - \|u_\rho^x\|] > 0.
\end{aligned}$$

If for given x we take $y = \lambda_x u_\rho^x$ we have that g satisfies condition (θ) with respect to $\mathbf{K}$. Because we can show that g is continuous, applying *Theorem 2* we deduce that the problem $NCP(g, \mathbf{K})$ has a solution $x_* \in \mathbf{K}$. The solution x_* is such that $\|x_*\| \leqslant \rho$. Indeed, if $\|x_*\| > \rho$ we must have

$$\langle x_* - \lambda_{x_*} u_\rho^{x_*}, g(x_*)\rangle > 0 \tag{19}$$

or

$$\langle \lambda_{x_*} u_\rho^{x_*} - x_*, g(x_*)\rangle < 0 \tag{20}$$

which is impossible, because the problem $NCP(g, \mathbf{K})$ being equivalent to a variational inequality we have

$$\langle \lambda_{x_*} u_\rho^{x_*} - x_*, g(x_*)\rangle \geqslant 0$$

Hence, $\|x_*\| \leqslant \rho$ and in this case from the definition of g we have $g(x_*) = f(x_*)$, that is, x_* is a solution of the problem $NCP(f, \mathbf{K})$. □

The next result is close to *Theorem 6*. We say that a mapping $T : \mathbf{K} \to \mathbf{R}^n$ satisfies *condition* (β) if there exists a real number $\beta(T) > 0$ such that for all $x \in \mathbf{K}$ with $\|x\| \geqslant 1$ we have $\|T(x)\| \leqslant \beta(T)\|x\|$.

EXAMPLES

(1) Any linear continuous operator $T : \mathbf{R}^n \to \mathbf{R}^n$ satisfies condition (β).

(2) If $T : \mathbf{K} \to \mathbf{R}^n$ satisfies Lipschitz property, then T satisfies property (β). Indeed, let $x_0 \in \mathbf{K}$ a particular element. Since T is a Lipschitzian mapping, there exists $k > 0$ such that $\|T(x) - T(x_0)\| \leqslant k\|x - x_0\|$ for any $x \in \mathbf{K}$. It follows that $\|T(x)\| \leqslant \|T(x) - T(x_0)\| + \|T(x_0)\| \leqslant k\|x\| + k\|x_0\| + \|T(x_0)\|$. If we set $\beta_0 = k\|x_0\| + \|T(x_0)\|$ we have $\|T(x)\| \leqslant k\|x\| + \beta_0$, which implies for any x with $\|x\| \geqslant 1$ that $\|T(x)\| \leqslant k\|x\| + \beta_0\|x\| = (k + \beta_0)\|x\|$. Hence, T satisfies condition (β) with $\beta(T) = k + \beta_0$.

THEOREM 8. *Let $f : \mathbf{K} \to \mathbf{R}^n$ be a continuous function and $T : \mathbf{K} \to \mathbf{R}^n$ a mapping satisfying condition (β). If the following assumptions are satisfied:*

(1) $\lim_{\|x\|\to+\infty} < f(x) - T(x), x > /\|x\|^2 \geqslant k_0 > 0$,

(2) $\beta(T) < k_0$,

Then there exists $\rho > 1$ such that $\langle f(x), x\rangle > 0$ for all $x \in \mathbf{K}$ with $\|x\| > \rho$. Moreover, the problem $NCP(f, \mathbf{K})$ has a solution $x_* \in \mathbf{K}$ such that $\|x_*\| \leqslant \rho$.

Proof. Take an $\varepsilon > 0$ such that $\beta(T)+\varepsilon < k_0$. From assumption (1) there exists $\rho > 1$ such that for all $x \in \mathbf{K}$ with $\|x\| > \rho$ we have

$$\frac{\langle f(x) - T(x), x\rangle}{\|x\|^2} > k_0 - \varepsilon$$

which implies

$$\langle f(x) - T(x), x\rangle > (k_0 - \varepsilon)\|x\|^2$$

and finally,

$$\langle f(x), x\rangle > \langle T(x), x\rangle + (k_0 - \varepsilon)\|x\|^2.$$

From the last inequality we obtain
$\langle f(x), x\rangle \geqslant -\beta(T)\|x\|^2 + (k_0 - \varepsilon)\|x\|^2 = \|x\|^2(-\beta(T) - \varepsilon + k_0) > 0$ for all $x \in \mathbf{K}$ with $\|x\| > \rho$.

Applying *Theorem 6* we obtain that the problem $NCP(f, \mathbf{K})$ has a solution $x_* \in \mathbf{K}$. To finish, it is sufficient to observe that because $\langle f(x), x\rangle > 0$ for all $x \in \mathbf{K}$ with $\|x\| > \rho$, we must have $\|x_*\| \leqslant \rho$. □

REMARK. *Theorem 8* is applicable in the following two cases.

(a) $f(x) = T(x) + a\,x + b$, where $a > 0$, $b \in \mathbf{R}^n$ is an arbitrary vector and T satisfies condition (β) with $\beta(T) < a$.

(b) $f(x) = T(x) + Lx + b$, where $b \in \mathbf{R}^n$ is an arbitrary vector, L is a linear operator from $\mathbf{R}^n$ into $\mathbf{R}^n$ such that $\langle Lx, x\rangle \geqslant k_0\|x\|^2$ for any $x \in \mathbf{K}$ and T satisfies condition (β) with $\beta(T) < k_0$.

Also, *Theorem 8* has an interesting application to the Linear Complementarity Problem.

PROPOSITION 9. *Let A be an $n \times n$-matrix such that $A = A_1 + A_2$ and the following assumptions are satisfied:*

(1) $\langle A_2 x, x\rangle \geqslant k_0\|x^2\|$ *with* $k_0 > 0$ *for any* $x \in \mathbf{K}$,

(2) $\|A_1\| < k_0$, *where* $\|A_1\|$ *is the norm of* A_1 *considered as linear operator.*

Then the problem $LCP(A, b, \mathbf{K})$ has a solution for any $b \in \mathbf{R}^n$.

Proof. It is sufficient to remark that all assumptions of *Theorem 8* are satisfied for the mapping $f(x) = A_1x + A_2x + b$. □

4. Application to fixed point theory

Now, we apply *Theorem 7* to the *fixed point theory*. The next result is related to the classical *Altman's fixed point theorem* [1, 10].

THEOREM 10 (Generalization of Altman's Theorem). *If for the continuous mapping* $h : \mathbf{K} \to \mathbf{K}$ *there exists* $\rho > 0$ *such that for all* $x \in \mathbf{K}$ *with* $\|x\| = \rho$ *there exists* $u \in \mathbf{K}$ *with* $\|u\| < \rho$ *such that* $\langle x - u, x - h(x)\rangle \geqslant 0$, *then the mapping* h *has a fixed point in* $\mathbf{K}$.

Proof. From the complementarity theory it is known that the mapping $h : \mathbf{K} \to \mathbf{K}$ has a fixed point in $\mathbf{K}$ if and only if the problem $NCP(I - h, \mathbf{K})$ has a solution. Applying *Theorem 7,* the theorem follows. □

The next corollary can be considered as the analogue for cones of the well known *Altman's fixed point theorem* [1, 10].

COROLLARY 11. *If for the continuous function* $h : \mathbf{K} \to \mathbf{K}$ *there exists* $\rho > 0$ *such that for all* $x \in \mathbf{K}$ *with* $\|x\| = \rho$ *we have*

$$\|x\|^2 \geqslant \langle x, f(x)\rangle,$$

then h *has a fixed point in* $\mathbf{K}$.

REMARK. The assumption used in *Theorem 10* is more flexible than the assumption used in [10].

5. Complementarity problems with P_0-functions

Now, we will study the problem $NCP(f, R^n_+)$ associated with a P_0-function $f : \mathbf{R}^n_+ \to \mathbf{R}^n$. The class of P_0-function (P-function) were introduced by J.J. Moré and W. Rheinboldt [20] as a natural extension of the notion of a square matrix to be a P_0-matirix (P-matrix), i.e., if all its principal minors are nonnegative (positive).

We recall the definition. A function $f : D \subset \mathbf{R}^n \to \mathbf{R}^n$ is a P_0-function (P-function) on D if for all $x, y \in D, x \neq y$, there exists an index $i = i(x, y)$, such that $x_i \neq y_i$ and $(x_i - y_i)(f_i(x) - f_i(y)) \geqslant 0$, $((x_i - y_i)(f_i(x) - f_i(y)) > 0)$.

Considering the problem $NCP(f, \mathbf{R}^n_+)$, denote by $\mathcal{F}$ the set of all feasible solutions, i.e., $\mathcal{F} = \{x \in \mathbf{R}^n_+ | f_i(x) \geqslant 0,$ *for all* $i = 1, 2, \dots, n\}$. We say that $u \in \mathcal{F}$ is *strictly feasible* if $f_i(u) > 0$, for all $i = 1, 2, \dots, n$. In [19], J.J. Moré showed that if f is a monotone mapping, i.e., $\langle x - y, f(x) - f(y)\rangle \geqslant 0$ for all $x, y \in \mathbf{R}^n_+$) and $\mathcal{F}$ contains at least one strictly feasible point, then $NCP(f, \mathbf{R}^n_+)$ has a solution. This result cannot be extended to the class of P_0-functions (even P-function) as the following simple example shows. Let $f : \mathbf{R}^2 \to \mathbf{R}^2$ be defined by $f_1(x) = \psi(x_1) + x_2$ and $f_2(x) = x_2$, where $\psi : \mathbf{R} \to \mathbf{R}$ is

$$\psi(t) = -\frac{1}{2}\, e^{-t}.$$

The function f is a continuous P-function, and $\mathcal{F} = \{(x_1, x_2) \in \mathbf{R}^2_+ | x_1 \geqslant 0, x_2 \geqslant \frac{1}{2} e^{-x_1}\}$.

The point $u = (0, 1)$ is a strictly feasible point, but the only point which satisfies the complementarity condition associated with f is (0,0), which is not a solution of $NCP(f, \mathbf{R}^n_+)$ since $(0, 0) \notin \mathcal{F}$.

The next result shows that the problem $NCP(f, \mathbf{R}^n_+)$ associated to a P_0-function is solvable if the set $\mathcal{F}$ contains n points of a particular form.

THEOREM 12. *Let $f : \mathbf{R}^n_+ \to \mathbf{R}^n$ be a P_0-function. If the feasible set $\mathcal{F}$ contains n points $e^{(j)}$, $j = 1, 2, \ldots, n$ such that $e^{(j)}_j > 0$ and $e^{(j)}_i = 0$ for all $i \neq j$, then the problem $NCP(f, \mathbf{R}^n_+)$ has a solution.*

Proof. If, for a particular j, $f_j(e^{(j)}) = 0$ we have that $e^{(j)}$ is a solution of the problem $NCP(f, \mathbf{R}^n_+)$. Hence, we can suppose that for every $j \in \{1, 2, \ldots, n\}$ $f_j(e^{(j)}) > 0$. The theorem will be proved, if we show that f is without exceptional families of elements with respect to $\mathbf{R}^n_+$. Indeed, suppose that f has an exceptional family of elements $\{x^r\}_{r>0} \subset \mathbf{R}^n_+$. Because the particularities of the cone $\mathbf{R}^n_+$ we have for $\{x^r\}_{r>0}$ the following properties:

(i_1) $\|x^r\| \to +\infty$ as $r \to +\infty$,

(i_2) for every $r > 0$ there exists $\mu_r > 0$ such that

(a) $f_i(x^r) = -\mu_r x^r_i$, *if* $x^r_i > 0$,

(b) $f_i(x^r) \geqslant 0$, *if* $x^r_i = 0$.

By property (i_1) there exists an index $r > 0$ such that

$$\|x^r\| > \sqrt{\sum_{j=1}^{n} (e^{(j)}_j)^2}. \tag{21}$$

From (21) we have that there exists $j_0 \in \{1, 2, \ldots, n\}$ such that $x^r_{j_0} > e^{(j_0)}_{j_0}$. We observe that

$$x^r \neq e^{(j_0)} = (0, 0, \ldots, e^{(j_0)}_{j_0}, 0, 0, \ldots, 0). \tag{22}$$

Since f is a P_0-function, there exists $i = i(x^r, e^{(j_0)})$ such that $x^r_i \neq e^{(j_0)}_i$ and

$$(x^r_i - e^{(j_0)}_i)(f_i(x^r) - f_i(e^{(j_0)})) \geqslant 0. \tag{23}$$

If $i = j_0$ then we have

$$(x^r_{j_0} - e^{(j_0)}_{j_0})(f_{j_0}(x^r) - f_{j_0}(e^{(j_0)})) < 0.$$

which is a contradiction of (23).

If $i \neq j_0$ then we have $e^{(j_0)}_i = 0$ and $x^r_i > 0$, which imply again

$$(x^r_i - e^{(j_0)}_i)(f_i(x^r) - f_i(e^{(j_0)})) < 0.$$

The last inequality is also a contradiction of (23). We conclude that f is without exceptional families of elements with respect to $\mathbf{R}^n_+$, and by *Theorem 1* the problem $NCP(f, \mathbf{R}^n_+)$ has a solution. □

6. Application to the study of Generalized Linear Complementarity Problem

Theorem 12 can be used to extend to P_0-functions the main existence theorem for the *Generalized Linear Complementarity Problem* (known as the*Vertical Linear Complementarity Problem*)proved in [2].

We recall the definition of this problem. By a vertical block matrix M of type $(m_1, m_2, \ldots, m_n)$, we mean a matrix

$$M = \begin{pmatrix} M_1 \\ M_2 \\ \vdots \\ M_j \\ \vdots \\ M_n \end{pmatrix}$$

where the *j*-th block M^j has order $m_j \times n$. Thus for $m = \sum_{j=1}^n m_j$ the matrix M is of order $m \times n$. Let q be a vector in $\mathbf{R}^m$ partitioned conformably with M, i.e.,

$$q = \begin{pmatrix} q^1 \\ q^2 \\ \vdots \\ q^j \\ \vdots \\ q^n \end{pmatrix}$$

with $q^j \in \mathbf{R}^{m_j}$.

The Generalized Linear Complementarity Problem (associated with M and q), denoted by *GLCP*(M, q), is to find $z \in \mathbf{R}^n$ such that:

$$GLCP(M, q) : \begin{cases} z \geqslant 0,\ M^j z + q^j \geqslant 0_{m_j} \text{ and} \\ z_j \prod_{i=1}^{m_j} (M^j z + q^j)_i = 0 (j = 1, 2, \ldots, n) \end{cases}$$

where 0_{m_j} is the null vector in $\mathbf{R}^{m_j}$. This clearly agrees with the Linear Complementarity Problem when $m_j = 1$ and M^j is the *j*-th row of $M (j = 1, 2, \ldots, n)$. The problem *GLCP*(M, q) was defined in [4] and it has been studied recently in [2, 5, 6, 17, 22–24].

We recall now some notions on rectangular matrices. Let M be a vertical block matrix of type $(m_1, m_2, \ldots, m_n)$. An $n \times n$ submatrix N of M is called a *representative submatrix* if its *j*-th row is drawn from the *j*-th block, M^j of M. The properties of M will be based upon properties of its representative submatrices. Having this concept, we can talk about principal submatrices of the rectangular matrix M. Obviously, a vertical block matrix M of type $(m_1, m_2, \ldots, m_n)$ has $\prod_{j=1}^{n} m_j$ representative submatrices.

Let M be a vertical block matrix of type $(m_1, m_2, \ldots, m_n)$. A *principal submatrix* of M is a principal submatrix of a representative submatrix of M. The determinant of such a matrix is a *principal minor* of M. A vertical block matrix M of type $(m_1, m_2, \ldots, m_n)$ is called a P_0-matrix (P-matrix) if and only if all its principal minors are nonnegative (strictly positive).

The next result is an existence theorem for the problem $GLCP(M, q)$ when M is a P_0-matrix.

THEOREM 13. *Let M be a P_0-vertical block matrix of type $(m_1, m_2, \ldots, m_n)$ and $q \in \mathbf{R}^m$ a vector partitioned conformably with M, $m = \sum_{j=1}^{n} m_j$. Assume that there exists n vectors $x^{(l)} = (x_k^l) l = 1, 2, \ldots, n$, $k = 1, 2, \ldots, n$ such that*

$$\begin{cases} \textit{for each } l = 1, 2, \ldots, n \\ x_k^l = 0 \textit{ for } k \neq l, x_l^l > 0 \textit{ and} \\ \min_{1 \leqslant i \leqslant m_j} \{(M^j x^{(l)})_i + q_i^j\} \geqslant 0, \ \textit{for } j = 1, 2, \ldots, n. \end{cases} \tag{24}$$

Then the problem GLCP(M, q) has a solution.

Proof. Consider the piecewise linear function $f : \mathbf{R}^n \to \mathbf{R}^n$ defined as

$$f_j(x) = \min_{l \leqslant i \leqslant m_j} \{(M^j x)_i + q_i^j\}, \ j = 1, 2, \ldots, n.$$

Clearly, the solvability of $GLCP(M, q)$ is equivalent to the solvability of the problem $NCP(f, \mathbf{R}_+^n)$. As already observed by A.A. Ebiefung [5], the assumption on M implies that f is a P_0-function, moreover condition (24) implies that the assumptions of *Theorem 12* hold for f defined above and $e^{(1)} = x^{(1)} \cdots e^{(n)} = x^{(n)}$. Hence the result follows from *Theorem 12.* □

REMARK. If M is a P-vertical block matrix this problem has been solved in [4] in the case that the feasible set is non-empty. It is known that if M is a P-matrix, then the solution is unique [19]. With a different technique, in 1989, in his Ph.D. Thesis, B. P. Szanc [22] proved that the existence result, for a P-vertical block matrix, follows from the fact that the function f is a non degenerate P-function. *Theorem 13* is a more general result.

Acknowledgments

Thanks for Professor I. Massabo from University of Calabria for the numberless stimulating conversations about the subject of this paper.

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